On the Complexity of Sum-of-Products Problems over Semirings

Thomas Eiter, Rafael Kiesel

Vienna University of Technology

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LOGICAL METHODS IN COMPUTER SCIENCE

Sum-of-Products [Bacchus *et al.*, 2009]

- given a finite domain $\mathcal{D}$ and
- functions $f_i : \mathcal{D}^{n_i} \to \mathbb{R}$ $(i = 1, \ldots, n)$
- compute

$$\sum_{X_1,\ldots,X_m \in \mathcal{D}} \prod_{i=1}^{n} f_i(\vec{Y}_i),$$

- where $\vec{Y}_i$ is a vector of variables from $\{X_1, \ldots, X_m\}$.

Sum-of-Products over Semirings [Bacchus *et al.*, 2009]

- More generally, over some *semiring* $\mathcal{R} = (R, \oplus, \otimes, e_\oplus, e_\otimes)$:
- given a finite domain $\mathcal{D}$ and
- functions $f_i : \mathcal{D}^{n_i} \to R$ $(i = 1, \dots, n)$
- compute

$$\bigoplus_{X_1,\dots,X_m \in \mathcal{D}} \bigotimes_{i=1}^{n} f_i(\vec{Y}_i),$$

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Semirings

A semiring is an algebraic structure $(R, \oplus, \otimes, e_\oplus, e_\otimes)$, s.t.

- $(R, \oplus, e_\oplus)$ is a commutative monoid with neutral element $e_\oplus$
- $(R, \otimes, e_\otimes)$ is a monoid with neutral element $e_\otimes$
- multiplication ($\otimes$) distributes over addition ($\oplus$)
- multiplication by $e_\oplus$ annihilates R

Semiring Examples

Prominent examples are

- $\mathbb{B} = (\{\bot, \top\}, \vee, \wedge, \bot, \top)$ boolean
- $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ natural numbers
- $\mathcal{R}_{max} = (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$ max-plus

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$$\max_{a_1,\ldots,a_n \in \{0,1\}} \sum_{j=1}^{m} w_j \mathbb{1}_{C_j}$$

Semirings in Computer Science

Semirings were successfully used to parameterize “calculation” in

- Semiring-based Constraint Satisfaction Problems
  [Bistarelli *et al.*, 1999]
- Provenance
  [Green *et al.*, 2007]
- Semiring-based Argumentation
  [Bistarelli and Santini, 2010]
- Algebraic Model Counting
  [Kimmig *et al.*, 2017]
- Algebraic Constraints in Answer Set Programming
  [Eiter and Kiesel, 2020]

Complexity?

Known results:

- Completeness results for some specific semirings
  - #P-complete over $\mathbb{N}$
  - NP-complete over $\mathbb{B}$
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$\hookrightarrow$ Need a more in-depth complexity analysis!

Weighted Propositional Formulas

- Let $\mathcal{R} = (R, \oplus, \otimes, e_\oplus, e_\otimes)$ be a semiring
- Syntax

$$\alpha ::= k \mid v \mid \neg v \mid \alpha + \alpha \mid \alpha * \alpha,$$

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- Semantics of α given interpretation $\mathcal{I}$

$$\llbracket k \rrbracket_{\mathcal{R}}(\mathcal{I}) = k \quad (k \in R)$$

$$\llbracket \ell \rrbracket_{\mathcal{R}}(\mathcal{I}) = \begin{cases} e_{\otimes} & \ell \in \mathcal{I} \\ e_{\oplus} & \text{otherwise.} \end{cases} \quad (\ell \in \{v, \neg v\})$$

$$\llbracket \alpha_1 + \alpha_2 \rrbracket_{\mathcal{R}}(\mathcal{I}) = \llbracket \alpha_1 \rrbracket_{\mathcal{R}}(\mathcal{I}) \oplus \llbracket \alpha_2 \rrbracket_{\mathcal{R}}(\mathcal{I})$$

$$\llbracket \alpha_1 * \alpha_2 \rrbracket_{\mathcal{R}}(\mathcal{I}) = \llbracket \alpha_1 \rrbracket_{\mathcal{R}}(\mathcal{I}) \otimes \llbracket \alpha_2 \rrbracket_{\mathcal{R}}(\mathcal{I})$$

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SAT($\mathcal{R}$):

Given a weighted formula α over variables in $\mathcal{V}$ compute

$$\bigoplus_{\mathcal{I} \in \mathsf{Int}(\mathcal{V})} [\![\alpha]\!]_{\mathcal{R}}(\mathcal{I})$$

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  $\delta \subseteq (Q \times (\Sigma \cup R)) \times (Q \times (\Sigma \cup R)) \times \{-1, 1\} \times R$[1]

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     ($r \in R'$ or $r = \sigma_1 \in R$)
  3. *cannot differentiate semiring values*
     ($\sigma_1 \in R$ implies that for all $\sigma_1' \in R$ we have $((q_1, \sigma_1'),$ $(q_2, \sigma_1'), e, r') \in \delta$, where $r' = \sigma_1'$ if $r = \sigma_1$ and else $r' = r$)

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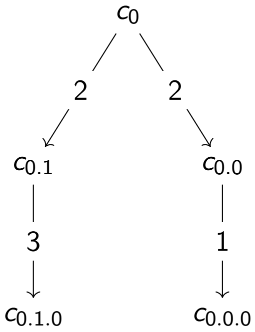
SRTM Output

- Let M be an SRTM and $c = (q, w, n)$ a configuration, where q is a state, w is the string on the tape and n is the head position



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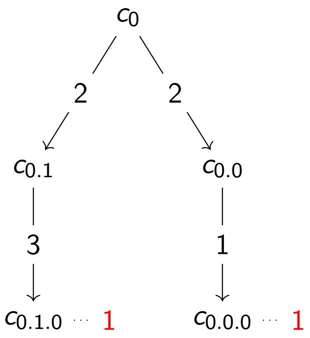

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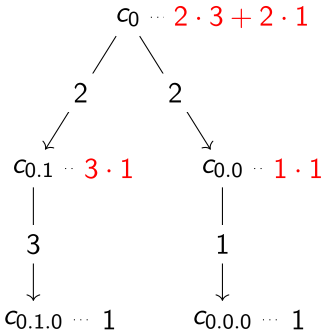

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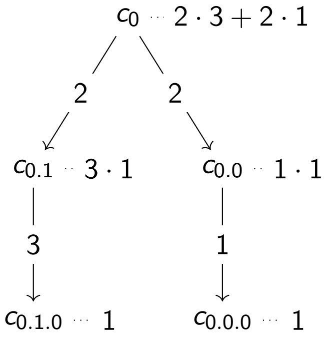
- The output is $v(c_0)$, the value of the initial configuration c_0.



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Theorem

SAT($\mathcal{R}$) *is* NP($\mathcal{R}$)*-complete with respect to polynomial transformations*[2]*, for every semiring* $\mathcal{R}$*.*

[2] i.e. the same kind we use for NP-completeness

NP($\mathcal{R}$)-complete Problems

The following problems are NP($\mathcal{R}$)-complete by reduction from SAT($\mathcal{R}$):

- Sum-of-Products

  [Bacchus *et al.*, 2009]
- Semiring-based Constraint Satisfaction Problems

  [Bistarelli *et al.*, 1999]
- Algebraic Model Counting

  [Kimmig *et al.*, 2017]
- Algebraic Constraint Evaluation

  [Eiter and Kiesel, 2020]

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Example

The binary representation $bin(n) = b_0 \ldots b_m$ such that $n = \sum_{i=1}^{m} b_i 2^i$ is an encoding function for the semiring $\mathbb{N}$ of the natural numbers

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- Unary encoding: Knapsack is in P
- There is a semiring whose multiplication is *undecidable* or *linear time* depending on the encoding

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- 1. and 2. are orthogonal
  $\hookrightarrow$ we consider 2.

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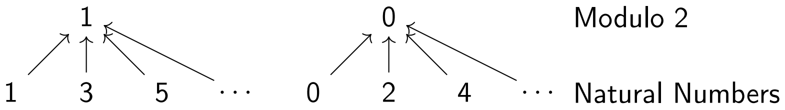

Epimorphism Theorem

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Theorem

Let $e_i(\mathcal{R}_i), i = 1, 2$ *be two encoded semirings, such that*

1. $\mathsf{SAT}(e_1(\mathcal{R}_1))$ *is in* FPSpace(poly),
2. *there exists a polynomial time computable epimorphism* $f : e_1(R_1) \rightarrow e_2(R_2)$, *and*
3. *for each* $e_2(r_2) \in e(R_2)$ *one can compute in polynomial time* $e_1(r_1)$ *s.t.* $f(e_1(r_1)) = e_2(r_2)$ *from* $e_2(r_2)$.

Then $\mathsf{SAT}(e_2(\mathcal{R}_2))$ *is counting-reducible to* $\mathsf{SAT}(e_1(\mathcal{R}_1))$.

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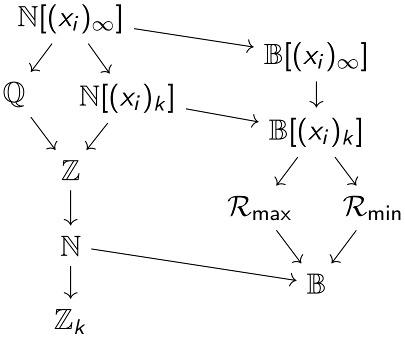

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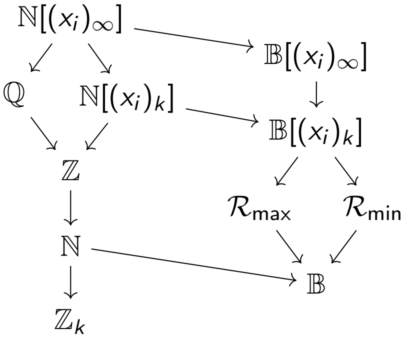

- Note: $\mathbb{N}[(x_i)_\infty], \mathbb{B}[(x_i)_\infty]$ have epimorphisms to every commutative countable (resp. and idempotent) semiring

Negative Results

Theorem

Let $\mathcal{R} = \mathbb{N}[(x_i)_\infty]$ (resp. $\mathcal{R} = \mathbb{B}[(x_i)_\infty]$).
The following are equivalent:

1. *There is an encoding function e for $\mathcal{R}$ s.t.*
   1) $\|[\![\alpha]\!]_\mathcal{R}(\mathcal{I})\|_e$ *is polynomial in the size of* α, $\mathcal{I}$,
   2) *we can extract the coefficient of* $x_{i_1}^{j_1} \ldots x_{i_n}^{j_n}$ *from* $e(r)$ *in polynomial time in* $\|r\|_e$, *and*
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- link to open complexity theoretic questions!

Positive Results

Theorem

Let e be the encoding function that represents exponents in unary and coefficients in binary. Then

- $\mathsf{SAT}(e(\mathbb{Q}[(x_i)_k]))$ *is counting-reducible to* #SAT *and* #P-*hard for counting reductions.*
- $\mathsf{SAT}(e(\mathbb{B}[(x_i)_k]))$ *is* $\mathsf{FP}^{\mathsf{NP}}_{\|}$*-complete for metric reductions.*

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$\hookrightarrow$ $\mathcal{R}$ idempotent $\rightarrow$ epimorphism from $\mathbb{B}[(x_i)_n]$

- Solve SAT($\mathcal{R}$) via SAT($\mathbb{N}[(x_i)_n]$) (resp. SAT($\mathbb{B}[(x_i)_n]$))

Conclusion

- SAT($\mathcal{R}$) and NP($\mathcal{R}$) are quantitative semiring counterparts to SAT and NP

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- Results suggest that the complexity strongly depends on the amount of information retained
- In general SAT($\mathcal{R}$) harder than #SAT (unless $\#P \subseteq FP/poly$)
- For a broad class of commutative, finitely generated semirings SAT($\mathcal{R}$) can be reduced to #SAT (and is in $FP_{\parallel}^{NP}$ if $\mathcal{R}$ is also idempotent)

Fahiem Bacchus, Shannon Dalmao, and Toniann Pitassi.
Solving# sat and bayesian inference with backtracking search.
JAIR, 34:391–442, 2009.

Stefano Bistarelli and Francesco Santini.
A common computational framework for semiring-based argumentation systems1, 2.
In *ECAI*, volume 215, page 131, 2010.

Stefano Bistarelli, Ugo Montanari, Francesca Rossi, Thomas Schiex, Gérard Verfaillie, and Hélene Fargier.
Semiring-based csps and valued csps: Frameworks, properties, and comparison.
Constraints, 4(3):199–240, 1999.

Thomas Eiter and Rafael Kiesel.
Asp(ac): Answer set programming with algebraic constraints.
arXiv preprint arXiv:2008.04008, 2020.

Todd J Green, Grigoris Karvounarakis, and Val Tannen.
Provenance semirings.
In *Proceedings of the twenty-sixth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 31–40. ACM, 2007.

Angelika Kimmig, Guy Van den Broeck, and Luc De Raedt.
Algebraic model counting.
Journal of Applied Logic, 22:46–62, 2017.